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Symmetry and Classification of Certain Regular Group Divisible Designs

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Abstract

In a regular group divisible (GD) design with parameters v, b, r, k, λ_1, λ_2 satisfying $rk - \lambda_2 v = 1$ and $\lambda_2 = \lambda_1 + 1$, it is shown that the design must be symmetrical (i.e., v = b). Furthermore, the parameters of such symmetrical regular GD designs can be derived in terms of only two integral parameters.

1. Introduction

The largest, simplest and perhaps most important class of 2-associate partially balanced incomplete block designs is known as group divisible (GD). A GD design is an arrangement of v (=mn) treatments in b blocks such that (i) each block contains k distinct treatments, k < v; (ii) each treatment is replicated r times; and (iii) the mn treatments can be divided into m groups of n treatments each, any two treatments occurring together in λ_1 blocks if they belong to the same group, and in λ_2 blocks if they belong to different groups. GD designs are classified into three subtypes : (a) singular, if $r = \lambda_1$; (b) semi-regular, if $r > \lambda_1$ and $rk = \lambda_2 v$; (c) regular, if $r > \lambda_1$ and $rk > \lambda_2 v$

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In this paper, as some boundary between a semi-regular type and a regular type, we may consider a regular design satisfying $rk-\lambda_2 v=1$. Among this class, a case with $\lambda_2=\lambda_1+1$, that has a strong statistical meaning on optimality, is discussed. We here characterize regular GD designs with parameters v, b, r, k, λ_1, λ_2 satisfying $rk-\lambda_2 v = 1$ and $\lambda_2 = \lambda_1+1$, by showing a symmetry of the design and deriving a presentation of parameters in terms of only two positive integers.

2. Symmetry of Regular Designs

The problem in this section is now as follows. Does there exist a non-trivial regular GD design with parameters $v = mn$, b, r, k, λ_1, λ_2 satisfying

(2.1) $$rk - \lambda_2 v = 1,$$

(2.2) $$\lambda_2 = \lambda_1 + 1,$$

(2.3) $$b > v \ ?$$

To show the non-existence of such GD designs, we first need two lemmas.

Lemma 2.1. Let β, γ, δ be integers such that $\beta, \gamma \geqq 2$, $0 < \delta < \beta\gamma$ and $(\beta\gamma^2-1)/\{\delta(\beta\gamma-\delta)\}$ is an integer greater than or equal to 2. Then either $1 < \delta < \gamma$ or $(\beta-1)\gamma < \delta < \beta\gamma-1$.

Proof. Since $\beta, \gamma \geqq 2$, it is obvious that $(\beta\gamma^2-1)/\{\delta(\beta\gamma-\delta)\}$ cannot be an integer if $\delta = 1$ or $\delta = \beta\gamma-1$. Hence $1 < \delta < \beta\gamma-1$. Now

(2.4) $(\beta\gamma^2-1)/\{\delta(\beta\gamma-\delta)\} \geqq 2$ iff $2\delta^2 - 2\beta\gamma\delta + (\beta\gamma^2-1) \geqq 0$.

The roots of the equation $2\delta^2 - 2\beta\gamma\delta + (\beta\gamma^2-1) = 0$ on δ are given by

$$\delta = (\beta\gamma)/2 \pm (1/2)(\gamma^2\beta(\beta-2)+2)^{1/2}.$$

Hence in order that (2.4) holds it is necessary that

$$|\delta - (\beta\gamma)/2| \geqq (1/2)(\gamma^2\beta(\beta-2)+2)^{1/2} > (1/2)(\gamma^2(\beta-2)^2)^{1/2}$$

$= \gamma(\beta-2)/2,$

i.e., either $\delta > \gamma(\beta-1)$ or $\delta < \gamma$. Thus, either $1 < \delta < \gamma$ or $\gamma(\beta-1) < \delta < \beta\gamma-1$, completing the proof.

Lemma 2.2. Let β, γ, δ ($\geqq 2$) be integers such that $(\beta\gamma^2-1)/\{\delta(\beta\gamma-\delta)\}$ ($=\xi$, say) is an integer and $\delta < \gamma$. Further let $\gamma = \mu\delta - \alpha$ where $0 \leqq \alpha \leqq \delta-1$. Then $\xi = \mu$.

Proof. First note that if $(\beta\gamma^2-1)/\{\delta(\beta\gamma-\delta)\}$ is an integer then so is $(\beta\gamma^2-1)/\delta$. Therefore, γ cannot be an integral multiple of δ, i.e., $1 \leqq \alpha \leqq \delta-1$. Now

$$(2.5)\quad \xi = (\beta\gamma^2-1)/\{\delta(\beta\gamma-\delta)\} = [\beta\gamma\{(\mu-1)\delta + \delta - \alpha\} - 1]/(\beta\gamma\delta-\delta^2)$$
$$= (\mu-1) + \{(\mu-1)\delta^2 + \beta\gamma(\delta-\alpha) - 1\}/(\beta\gamma\delta-\delta^2).$$

Since $\gamma > \delta$, we have $\mu \geqq 2$ and the second term in the right-hand side of (2.5) must be positive. Therefore $\xi > \mu-1$. Since ξ is an integer, we get

$$(2.6)\qquad \xi \geqq \mu.$$

Since $\mu \geqq 2$, it follows that $\xi \geqq 2$. Next,

$$(\beta\gamma^2-1)/\{\delta(\beta\gamma-\delta)\} = \xi \text{ iff } \beta\gamma^2-1 = \xi\delta\beta\gamma-\xi\delta^2 \text{ iff } \beta = (\xi\delta^2-1)/\{\gamma(\xi\delta-\gamma)\}$$

Hence $2 \leqq \gamma < \xi\delta$ and by Lemma 2.1 (replacing β, γ, δ there by ξ, δ, γ respectively), it follows that either $\gamma < \delta$ or $\gamma > (\xi-1)\delta$. Since $\gamma > \delta$, we obtain $\gamma > (\xi-1)\delta$. Recalling that $\gamma = \mu\delta-\alpha$, it follows that $\mu\delta-\alpha > (\xi-1)\delta$, i.e., $(\mu-\xi+1)\delta > \alpha$. Since $\alpha \geqq 1$, we must have $\mu-\xi+1 \geqq 1$, i.e., $\mu \geqq \xi$ and applying (2.6), the proof is completed.

Consider now the main problem. Let $m, n \geqq 2$ in $v = mn$ for non-triviality. Now, a well-known relation $r(k-1) = (n-1)\lambda_1+n(m-1)\lambda_2$ with (2.1) and (2.2) implies

$$(2.7)\qquad r = n + \lambda_2.$$

Hence by (2.1)

(2.8) $k = (v\lambda_2+1)/r = (mn\lambda_2+1)/(n+\lambda_2) = mn - (mn^2-1)/(n+\lambda_2)$

which shows that $(mn^2-1)/(n+\lambda_2)$ must be an integer, say t (> 0). Obviously

(2.9) $\lambda_2 = (mn^2-1)/t - n.$

The relation $bk = vr$ yields, applying (2.7) and (2.8),

(2.10) $b = vr/k = mn(n+\lambda_2)^2 / (mn\lambda_2+1).$

Note that mn and $mn\lambda_2+1$ are relatively prime and b is an integer. Therefore, by (2.10), $(n+\lambda_2)^2/(mn\lambda_2+1)$ must be an integer. Also by (2.3) and (2.10), $(n+\lambda_2)^2/(mn\lambda_2+1)$ is an integer greater than or equal to 2. By (2.9), after some simplification,

$$(n+\lambda_2)^2/(mn\lambda_2+1) = (mn^2-1)/\{t(mn-t)\}$$

and hence $(mn^2-1)/\{t(mn-t)\}$ must be an integer ($\geqq 2$). Obviously $0 < t < mn$. Applying Lemma 2.1, we have established the following.

Theorem 2.1. For the existence of a regular GD design satisfying (2.1), (2.2) and (2.3), it is necessary that there exist positive integers m, n, t such that $m, n \geqq 2$, $(mn^2-1)/\{t(mn-t)\}$ is an integer and either $1 < t < n$ or $(m-1)n < t < mn-1$.

If a combination, m_0, n_0, t_0, say, satisfies the conditions of Theorem 2.1 with $(m_0-1)n_0 < t_0 < m_0n_0-1$, then the combination m_0, n_0, t_0' also satisfies the same conditions where $t_0' = m_0n_0-t_0$, and obviously $1 < t_0' < n_0$. Hence we obtain the following.

Theorem 2.1A. For the existence of a regular GD design satisfying (2.1), (2.2) and (2.3), it is necessary that there exist positive integers m, n, t such that $m, n, t \geqq 2$, $(mn^2-1)/\{t(mn-t)\}$ is an integer and $t < n$.

In view of Theorem 2.1A, in order to establish the non-existence of a regular GD design satisfying (2.1), (2.2) and (2.3), it will be enough to prove the following.

<u>Theorem 2.2</u>. There do not exist integers m, n, t ($\geqq 2$) such that $(mn^2-1)/\{t(mn-t)\}$ is an integer and $t < n$.

<u>Proof</u>. If possible suppose there exist integers m, n, t ($\geqq 2$) such that $(mn^2-1)/\{t(mn-t)\}$ is an integer and $t < n$. Then $(mn^2-1)/t$ is an integer and hence n must be relatively prime to t. Therefore, there exist a positive integer g and integers $\alpha_1, \alpha_2, \cdots, \alpha_{g-1}$, $\mu_1, \mu_2, \cdots, \mu_g$ such that

$$(2.11)\qquad \begin{aligned} n &= \mu_1 t - \alpha_1, \\ t &= \mu_2\alpha_1 - \alpha_2, \\ \alpha_1 &= \mu_3\alpha_2 - \alpha_3, \\ \alpha_2 &= \mu_4\alpha_3 - \alpha_4, \\ &\ \ \vdots \\ \alpha_{g-3} &= \mu_{g-1}\alpha_{g-2} - \alpha_{g-1}, \\ \alpha_{g-2} &= \mu_g\alpha_{g-1} - 1, \end{aligned}$$

and

$$(2.12)\qquad t > \alpha_1 > \alpha_2 > \alpha_3 > \cdots > \alpha_{g-2} > \alpha_{g-1} > 1.$$

The relation (2.12), together with the fact $n > t$, shows that

$$(2.13)\qquad \mu_1, \mu_2, \cdots, \mu_g \geqq 2.$$

Now, by Lemma 2.2 and the first relation in (2.11),

$$(mn^2-1)/\{t(mn-t)\} = \mu_1,$$

i.e., $\mu_1 = (mn^2-1)/\{t(mn-t)\} = \{mn(\mu_1 t-\alpha_1)-1\}/(mnt-t^2)$

$$(2.14)\qquad = \mu_1 + \{\mu_1 t^2 - mn\alpha_1 - 1\}/(mnt-t^2)$$

which shows that $\mu_1 t^2 - mn\alpha_1 - 1 = 0$ and then $m = (\mu_1 t^2-1)/(n\alpha_1)$

$= (\mu_1 t^2-1)/\{\alpha_1(\mu_1 t-\alpha_1)\}$. Since m is an integer, it follows from (2.12), (2.13), the second relation in (2.11) and Lemma 2.2 that

$$(2.15) \qquad (\mu_1 t^2-1)/\{\alpha_1(\mu_1 t-\alpha_1)\} = \mu_2.$$

Proceeding as in (2.14), it follows from (2.15) that $\mu_2\alpha_1^2 - \mu_1 t\alpha_2 - 1 = 0$ and hence $\mu_1 = (\mu_2\alpha_1^2 - 1)/(t\alpha_2) = (\mu_2\alpha_1^2 - 1)/\{\alpha_2(\mu_2\alpha_1-\alpha_2)\}$. Since μ_1 is an integer, it follows from (2.12), (2.13), the third relation in (2.11) and Lemma 2.2 that

$$(2.16) \qquad (\mu_2\alpha_1^2-1)/\{\alpha_2(\mu_2\alpha_1-\alpha_2)\} = \mu_3.$$

Proceeding as in (2.14), we have from (2.16) that $\mu_3\alpha_2^2 - \mu_2\alpha_1\alpha_3 - 1 = 0$ and hence $\mu_2 = (\mu_3\alpha_2^2 - 1)/\{\alpha_3(\mu_3\alpha_2 - \alpha_3)\}$ which is an integer. Thus, proceeding step by step, consider the last but one relation in (2.11) to obtain that $(\mu_{g-1}\alpha_{g-2}^2 - 1)/\{\alpha_{g-1}(\mu_{g-1}\alpha_{g-2} - \alpha_{g-1})\}$ is an integer. From (2.12), (2.13) and Lemma 2.2, since $\alpha_{g-2} = \mu_g\alpha_{g-1}-1$, it follows that

$$(\mu_{g-1}\alpha_{g-2}^2 - 1)/\{\alpha_{g-1}(\mu_{g-1}\alpha_{g-2} - \alpha_{g-1})\} = \mu_g,$$

whence proceeding as in (2.14) and using the last relation in (2.11) we get $\mu_g\alpha_{g-1}^2 - \mu_{g-1}\alpha_{g-2} - 1 = 0$ which implies

$$(2.17) \qquad \mu_{g-1} = (\mu_g\alpha_{g-1}^2-1)/\alpha_{g-2} = (\mu_g\alpha_{g-1}^2-1)/(\mu_g\alpha_{g-1}-1)$$
$$= \alpha_{g-1} + (\alpha_{g-1}-1)/(\mu_g\alpha_{g-1}-1).$$

Since by (2.12), (2.13), $\alpha_{g-1}, \mu_g \geqq 2$, the second term in the right-hand side of (2.17) is a proper fraction. Therefore, from (2.17), μ_{g-1} is not an integer. This is a contradiction. Thus, the proof is completed.

Thus, a regular GD design with parameters v, b, r, k, λ_1, λ_2 satisfying $rk-\lambda_2 v = 1$ and $\lambda_2 = \lambda_1+1$ must be symmetrical (i.e.,

$v = b$).

Remark 2.1. The following is an illustration of the relation (2.11): Let $n = 32$, $t = 25$. Then

$$\begin{aligned} 32 &= 2\cdot 25 - 18, \\ 25 &= 2\cdot 18 - 11, \\ 18 &= 2\cdot 11 - 4, \\ 11 &= 3\cdot 4 - 1. \end{aligned}$$

Hence $g = 4$, $\alpha_1 = 18$, $\alpha_2 = 11$, $\alpha_3 = 4$, $\mu_1 = 2$, $\mu_2 = 2$, $\mu_3 = 2$, $\mu_4 = 3$.

Remark 2.2. From Theorems 2.1 and 2.2, it holds that: There do not exist integers m, n, t ($\geqq 2$) such that $(mn^2-1)/\{t(mn-t)\}$ is an integer greater than or equal to 2. Writing $h = (mn^2-1)/\{t(mn-t)\}$, it follows that there do not exist integers m, n, t, h ($\geqq 2$) such that

(2.18) $$mn^2 - mhnt + ht^2 = 1.$$

It is interesting to see whether the non-existence of m, n, t, h satisfying (2.18) follows directly from the arithmetic theory of quadratic forms (treating m and h temporarily fixed) or from the theory of Diophantine equations. In that case the proofs for the present problem can perhaps be considerably simplified. But the authors do not know such an approach.

3. Classification of Regular Designs

We shall derive the parameters of regular GD designs discussed in Section 2 in terms of only two integral parameters.

Lemma 3.1. Let a_1, a_2 be integers satisfying $a_1, a_2 > 1$. Suppose

$$(a_1^2-1)/a_2 = a_3 \text{ (say) and } (a_2^2-1)/a_1 = a_4 \text{ (say)}$$

are integers. Then each of $(a_1^2-1)/a_3$, $(a_3^2-1)/a_1$, $(a_2^2-1)/a_4$ and $(a_4^2-1)/a_2$ is an integer.

Proof. Trivially $(a_1^2-1)/a_3 = a_2$ and $(a_2^2-1)/a_4 = a_1$ are integers. Now, since $a_2 = (a_1^2-1)/a_3$, we have

$$a_4 = (a_2^2-1)/a_1 = \{(a_1^2-1)^2-a_3^2\}/(a_1a_3^2)$$
$$= [(a_1-1)\{(a_1-1)(a_1+1)^2+a_3^2\}]/(a_1a_3^2) - 1.$$

Since a_4 is an integer, it follows that

$$[(a_1-1)\{(a_1-1)(a_1+1)^2+a_3^2\}]/(a_1a_3^2)$$

is an integer and in particular, $(a_1-1)\{(a_1-1)(a_1+1)^2+a_3^2\}/a_1$ is an integer. Since $a_1 > 1$, a_1 and a_1-1 must be relatively prime. Consequently, $\{(a_1-1)(a_1+1)^2+a_3^2\}/a_1$ is an integer. Now,

$$(a_1-1)(a_1+1)^2+a_3^2 = a_1^3 + a_1^2 - a_1 + (a_3^2-1),$$

and, therefore, the integrality of $\{(a_1-1)(a_1+1)^2+a_3^2\}/a_1$ shows that $(a_3^2-1)/a_1$ is an integer. Similarly, $(a_4^2-1)/a_2$ is also an integer. Thus the proof is completed.

Let α be a positive integer with $\alpha \geqq 2$. Define polynomials $P_t(\alpha)$ for $t \geqq 0$ as

$$(3.1) \qquad \begin{aligned} &P_0(\alpha) = 1, \quad P_1(\alpha) = \alpha, \\ &P_t(\alpha) = \alpha P_{t-1}(\alpha) - P_{t-2}(\alpha), \quad t = 2,3,\cdots. \end{aligned}$$

In this case, it is easy to show the following two lemmas.

Lemma 3.2. For each fixed α ($\geqq 2$), $\{P_t(\alpha)\}$, $t \geqq 0$, is a strictly increasing sequence of positive integers.

Remark 3.1. From Lemma 3.2, it is obvious that $P_t(\alpha) > 0$ for all $t \geqq 0$ and $\alpha \geqq 2$. By (3.1), $P_0(\alpha) = 1$, $P_1(\alpha) = \alpha$, $P_2(\alpha) = \alpha^2-1$, $P_3(\alpha) = \alpha^3-2\alpha$, $P_4(\alpha) = \alpha^4-3\alpha^2+1$, etc.

Lemma 3.3. For each t ($\geqq 2$) and each α ($\geqq 2$),

$$\{P_t(\alpha) + P_{t-2}(\alpha)\}/P_{t-1}(\alpha) = \alpha.$$

In particular, $P_t(2) = t+1$ for every t.

On the other hand, it follows that the definition in (3.1) gives another representation of $P_t(\alpha)$ as

$$(3.2)\quad \begin{cases} P_0(\alpha) = 1, \quad P_1(\alpha) = \alpha, \\ P_{t-2}(\alpha)P_t(\alpha) = P_{t-1}(\alpha)^2 - 1, \ t = 2,3,\cdots. \end{cases}$$

In fact, $P_{t-2}(\alpha)P_t(\alpha) - P_{t-1}(\alpha)^2 = P_{t-2}(\alpha)\{\alpha P_{t-1}(\alpha)-P_{t-2}(\alpha)\}-P_{t-1}(\alpha)^2$ $= \alpha P_{t-2}(\alpha)P_{t-1}(\alpha)-P_{t-2}(\alpha)^2-P_{t-1}(\alpha)^2 = P_{t-1}(\alpha)\{\alpha P_{t-2}(\alpha)-P_{t-1}(\alpha)\}-$ $P_{t-2}(\alpha)^2 = P_{t-1}(\alpha)\{P_{t-1}(\alpha)+P_{t-3}(\alpha)-P_{t-1}(\alpha)\}-P_{t-2}(\alpha)^2 = P_{t-1}(\alpha)P_{t-3}(\alpha)$ $-P_{t-2}(\alpha)^2$, for $t \geqq 3$. Hence $P_t(\alpha)P_{t-2}(\alpha)-P_{t-1}(\alpha)^2 = P_2(\alpha)P_0(\alpha)$ $-P_1(\alpha)^2 = -1$. [Note that Lemmas 3.2 and 3.3 can also be derived from (3.2) by induction.] The form of (3.2) will be used frequently.

Lemma 3.4. Let p and s be positive integers such that $p > s$. Then $(p^2-1)/s$ and $(s^2-1)/p$ are both integers if and only if p and s are of the form

$$p = P_t(\alpha), \qquad s = P_{t-1}(\alpha)$$

for some $\alpha \geqq 2$ and $t \geqq 1$.

Proof. [Sufficiency] Let $p = P_t(\alpha)$, $s = P_{t-1}(\alpha)$ for some $\alpha \geqq 2$ and $t \geqq 1$. Then by Lemma 3.2, $p > s$ and p, s are both positive integers. Now by (3.2), for $t \geqq 2$, $(p^2-1)/s = (P_t(\alpha)^2-1)/P_{t-1}(\alpha) = P_{t+1}(\alpha)$, $(s^2-1)/p = (P_{t-1}(\alpha)^2-1)/P_t(\alpha) = P_{t-2}(\alpha)$, which are both positive integers by Lemma 3.2. If $t = 1$, then $p = \alpha$, $s = 1$ and trivially $(p^2-1)/s = \alpha^2-1$ and $(s^2-1)/p = 0$ are integers. This proves the sufficiency part of the lemma. [Necessity] Let $p = \xi_0$, $s = \xi_1$ with $\xi_0 > \xi_1$. First suppose $\xi_1 = 1$, then $p = \xi_0 = P_1(\xi_0)$, $s = 1 = P_0(\xi_0)$, and the necessary part follows with $t = 1$ and $\alpha = \xi_0$. Next suppose $\xi_1 > 1$. Let

$$(3.3)\qquad \xi_2 = (s^2-1)/p = (\xi_1^2-1)/\xi_0.$$

Note that $\xi_0 > \xi_1$ implies $\xi_2 = (\xi_1^2-1)/\xi_0 \leqq (\xi_1^2-1)/(\xi_1+1) = \xi_1-1$, i.e.,

(3.4) $$\xi_2 < \xi_1.$$

Now, if $\xi_2 = 1$, then by (3.3), $\xi_0 = \xi_1^2 - 1$ and hence $\xi_0 = P_2(\xi_1)$, $\xi_1 = P_1(\xi_1)$ and the necessary part follows with $t = 2$ and $\alpha = \xi_1$. Next let $\xi_2 > 1$. Note that $(p^2-1)/s = (\xi_0^2-1)/\xi_1$ and $(s^2-1)/p = (\xi_1^2-1)/\xi_0$ $(= \xi_2)$ are both integers (with ξ_0, $\xi_1 > 1$) and hence by Lemma 3.1, $(\xi_1^2-1)/\xi_2$ and $(\xi_2^2-1)/\xi_1$ $(= \xi_3$, say) are both integers. Clearly, by (3.4), $\xi_3 \leqq (\xi_2^2-1)/(\xi_2+1) = \xi_2-1$, i.e.,

(3.5) $$\xi_3 < \xi_2 < \xi_1.$$

Now, if $\xi_3 = 1$, then $\xi_1 = \xi_2^2-1$, $\xi_0\xi_2 = \xi_1^2-1$, so that $\xi_0 = P_3(\xi_2)$, $\xi_1 = P_2(\xi_2)$ by (3.2). Consequently the necessary part follows again with $t = 3$ and $\alpha = \xi_2$. Following the above procedure (see in particular (3.5) - here $\{\xi_j\}$ is a decreasing sequence so that at some stage some ξ_j must be unity), in general there will exist integers g $(\geqq 1)$, $\xi_1 > \xi_2 > \cdots > \xi_g = 1$ such that $(\xi_1^2-1)/\xi_0 = \xi_2$, $(\xi_2^2-1)/\xi_1 = \xi_3$, $\cdots$, $(\xi_{g-2}^2-1)/\xi_{g-3} = \xi_{g-1}$, $(\xi_{g-1}^2-1)/\xi_{g-2} = 1$ $(=\xi_g)$. Then it follows from (3.2) that $p = \xi_0 = P_g(\xi_{g-1})$, $s = \xi_1 = P_{g-1}(\xi_{g-1})$, i.e., the necessary part holds with $t = g$ and $\alpha = \xi_{g-1}$ (> 1). This completes the proof of the lemma.

As seen in Section 2, the present design is symmetrical (i.e., $v = b$ and hence $r = k$). Consequently, after some manipulation along with $v = mn$, $rk-v\lambda_2 = 1$ and (2.7), the parameters of this design may be written as

(3.6)
$$\begin{aligned} &v = b = 2n + \lambda_2 + (n+1)(n-1)/\lambda_2, \\ &r = k = n + \lambda_2, \quad \lambda_1 = \lambda_2 - 1, \ \lambda_2; \\ &m = 2 + (n^2 + \lambda_2^2 - 1)/(n\lambda_2), \ n. \end{aligned}$$

Here, without loss of generality,

$$n > \lambda_2.$$

Because, the complement of the present design does not change both association scheme and eigenvalues corresponding to $r-\lambda_1$ and $rk-\lambda_2 v$ in the original design. Then we can consider only a case of $v \geqq 2k$ which implies $n > \lambda_2$.

Since m in (3.6) is an integer, $(n^2+\lambda_2^2 - 1)/(n\lambda_2)$ and then $(n^2 + \lambda_2^2 - 1)/n$ must be an integer. But $(n^2+\lambda_2^2-1)/n = n + (\lambda_2^2-1)/n$, so that $(\lambda_2^2-1)/n$ is an integer. Also the integrality of v shows that $(n^2-1)/\lambda_2$ is an integer. Thus, $(\lambda_2^2-1)/n$ and $(n^2-1)/\lambda_2$ are both integers and $n > \lambda_2$. Hence by Lemma 3.4, n and λ_2 are of the form

(3.7) $$n = P_t(\alpha), \quad \lambda_2 = P_{t-1}(\alpha)$$

for some $t \geqq 1$ and $\alpha \geqq 2$. Then

(3.8) $$(n^2-1)/\lambda_2 = (P_t(\alpha)^2-1)/P_{t-1}(\alpha) = P_{t+1}(\alpha),$$

while by Lemma 3.3,

(3.9) $$(n^2+\lambda_2^2-1)/(n\lambda_2) = \{(n^2-1)/\lambda_2 + \lambda_2\}/n$$
$$= \{P_{t+1}(\alpha)+P_{t-1}(\alpha)\}/P_t(\alpha) = \alpha.$$

Applying (3.7), (3.8), (3.9) and Lemma 3.3 to (3.6), the parameters of a regular GD design with $rk - \lambda_2 v = 1$ and $\lambda_2 = \lambda_1 + 1$ can be written as

(3.10) $$\begin{aligned} &v = b = (\alpha+2)P_t(\alpha), \quad r = k = P_t(\alpha) + P_{t-1}(\alpha), \\ &\lambda_1 = P_{t-1}(\alpha) - 1, \quad \lambda_2 = P_{t-1}(\alpha);\ m = \alpha + 2,\ n = P_t(\alpha), \end{aligned}$$

which mean that all the parameters may be expressed in terms of only two integral parameters α and t. For each α ($\geqq 2$), t ($\geqq 1$), $P_t(\alpha)$ is an integer by Lemma 3.2, so that the expressions in (3.10) must be integral-valued. Since $\alpha \geqq 2$, it follows from (3.10) that $m \geqq 4$ and $k \geqq 3$.

The existing regular GD designs with $rk - \lambda_2 v = 1$ and $\lambda_2 =$
+1 may be classified on the basis of (3.10) either according to
($\geqq$ 1) or according to α ($\geqq$ 2). For example,

) $t = 1$ leads to the series

$$v = b = \alpha(\alpha+2),\ r = k = \alpha+1,\ \lambda_1 = 0,\ \lambda_2 = 1,\ m = \alpha+2,\ n = \alpha;$$

i) $t = 2$ leads to the series

$$v = b = (\alpha+2)(\alpha^2-1),\ r = k = \alpha^2+\alpha-1,\ \lambda_1 = \alpha-1,\ \lambda_2 = \alpha,$$
$$m = \alpha+2,\quad n = \alpha^2-1;$$

ii) $\alpha = 2$ leads to the series

$$v = b = 4(t+1),\ r = k = 2t+1,\ \lambda_1 = t-1,\ \lambda_2 = t,\ m = 4,\ n = t+1.$$

ing a method in Theorem 8.6.2 of Raghavarao [3], it can be shown
at the existence of the series (i) is equivalent to the existence
an affine plane of order $\alpha+1$. Hence a design of the series (i)
th $\alpha = 5$ does not exist. As far as the authors know, all the
isting regular GD designs with $rk - \lambda_2 v = 1$ and $\lambda_2 = \lambda_1+1$ belong
one of the above three series.

mark 3.2. As shown above, for the existence of a regular GD design
th $rk-\lambda_2 v = 1$ and $\lambda_2 = \lambda_1+1$, it is necessary but not sufficient
at the parameters are of the form (3.10). This leads to a problem
identifying the values of t and α for which a GD design with the
rameters as in (3.10) really exists. But, this appears to be an
tremely hard problem. This is just analogous to the following
oblem: For an affine resolvable 2-design to exist it is necessary
at the parameters are of the form

$$v = \alpha^2\{(\alpha-1)t+1\},\quad b = \alpha\{\alpha^2 t+\alpha+1\},\quad r = \alpha^2 t+\alpha+1,$$
$$k = \alpha\{(\alpha-1)t+1\},\quad \lambda = \alpha t+1 \quad \text{for } \alpha \geqq 2 \text{ and } t \geqq 0.$$

it it is hard to identify the values of α and t for which such a

2-design actually exist (cf. Shrikhande [4]). Just similarly, it is extremely difficult to enumerate the values of α, t for which a GD design with parameters as in (3.10) really exists.

Finally, <u>all</u> the parameters of the present regular GD designs within the scope of parameters $r, k \leqq 10$ and $v \geqq 2k$ are listed along with references (cf. Clatworthy [1]).

No.	v = b	r = k	λ_1	λ_2	m	n	references
1	8	3	0	1	4	2	R54
2	15	4	0	1	5	3	R114
3	12	5	1	2	4	3	R145
4	24	5	0	1	6	4	R153
5	35	6	0	1	7	5	non-existence
6	16	7	2	3	4	4	John & Turner [2]
7	48	7	0	1	8	6	R183
8	63	8	0	1	9	7	R191
9	20	9	3	4	4	5	John & Turner [2]
10	80	9	0	1	10	8	R202
11	99	10	0	1	11	9	*unknown*

Note that the existence of a design of No. 11 is equivalent to the existence of an affine plane of order 10, which is unknown.

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References

. Clatworthy, W. H. : Tables of Two-Associate-Class Partially Balanced Designs. NBS Applied Mathematics Series 63, Washington, D. C. 1973.

. John, J. A., Turner, G. : Some new group divisible designs. J. Statist. Plann. Inf. 1, 103-107 (1977).

. Raghavarao, D. : Constructions and Combinatorial Problems in Design of Experiments. New York : John Wiley & Sons, Inc. 1971.

. Shrikhande, S. S. : Affine resolvable balanced incomplete block designs: a survey. Aequationes Math. 14, 251-269 (1976).

付記: 本論文は目下 "Graphs and Combinatorics" へ投稿中である.